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DEGENERATE r-STIRLING GENOCCHI POLYNOMIALS

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ABSTRACT. Degenerate exponential functions have extended the study of many types of polynomials. In particular, in this paper, we introduce the degenerate version of the r-Stirling Genocchi polynomials. This construction is based on the r-Stirling Genocchi polynomials, recently studied by Roberto and Bernard. This study aims to combine these polynomials in a degenerate exponential function to the mathematical understanding and explore new properties and potential applications of these extended polynomials.

1. INTRODUCTION

In [2], first of all, L. Carlitz studied the concept of degenerate exponential functions which many researchers have used to study various polynomials and numbers.

It is well known that the unsigned Stirling numbers of the first kind, denoted by ${n \brack k}$, count the number of permutations of the set $\{1,2,\dots,n\}$ with exactly k disjoint cycles. Similarly, the Stirling numbers of the second kind, denoted by ${n \brace k}$, count the number of ways to partition the set $\{1,2,\dots,n\}$ into k nonempty subsets (see[3],[4],[10]). From unsigned Stirling numbers of the first kind and Stirling numbers of the second kind, many researchers have used this to extend, especially the unsigned r-Stirling numbers of the first kind ${n+r \brack k+r}$ and r-Stirling numbers of the second kind ${n+r \brace k+r}$ (see[1],[5],[7],[9]). The degenerate unsigned r-Stirling numbers of the first kind ${n+r \brack k+r}_{r,\lambda}$ and the degenerate r-Stirling numbers of the second kind ${n+r \brace k+r}_{r,\lambda}$ have been studied by Kim-Kim-Lee-Park (see[7]). In [5], Roberto Corcino et al. studied SM r-Stirling numbers, and in an extension of this, in [6], Roberto-Vernard studied r-Stirling Genocchi polynomials. From r-Stirling Genocchi polynomials, we consider degenerate r-Stirling Genocchi polynomials. Based on previous studies, we focused on introducing the first kind degenerate r-Stirling Genocchi polynomial and the second kind degenerate r-Stirling Genocchi polynomial and obtaining their properties.

The outline of the paper is as follows. In section 1, we recall the degenerate exponential function and the degenerate logarithm function. we conjure unsigned degenerate Stirling numbers of the first kind and r-Stirling numbers of the first kind. We also remind the degenerate Stirling numbers of the second kind and r-Stirling numbers of the second kind. we recall Euler, Genocchi, Bernoulli polynomials and their higher-order α. In sections 2 and 3, we introduce the first kind degenerate r-Stirling Genocchi polynomials and the second kind degenerate r-Stirling Genocchi polynomials. In Theorem 1,4,10, we get known identities. In Theorem 2, we obtain an expression for $SG^1_{n,\lambda}(x,k,r)$. In Theorem 3, we get an expression for $SG^1_{n,\lambda}(x+1,k,r)$ with unsigned degenerate r-Stirling numbers of the first kind and degenerate Euler number. In Theorem 6, we have an expression of $SG^2_{n,\lambda}(x,k,r)$. In Theorem 7, from degenerate r-Stirling Genocchi polynomials, we get an

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identities $G_{m,\lambda}(x+r)\left\{{n-m \atop k}\right\}_\lambda = G_{m,\lambda}\left\{{n-m+r \atop k+r}\right\}_{r,\lambda}$. In Theorem 8, we get an expression of $2(k+1)\binom{n}{k+1}(r)_{n-(k+1),\lambda}$. In Theorem 9, we obtain identities for $SG^2_{n,\lambda}(x,k,r)$. In Theorem 11, we get an identities of $SG^{(2,\alpha)}_{n,\lambda}(x,k,r)$.

The degenerate exponential function defined by

$$e^x_\lambda(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see}[1,7-18]). \tag{1}$$

where $(x)_{0,\lambda}=1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n\geq 1)$ is degenerate falling factorial sequence.

The degenerate rising factorial sequence is defined by

$$<x>_{0,\lambda}=1 \quad <x>_{n,\lambda}=x(x-\lambda)\cdots(x+(n-1)\lambda), \quad (n\geq 1), \quad (\text{see}[18]). \tag{2}$$

The degenerate logarithm function is compositional inverse of $e_\lambda(t)$ which is defined by Kim-Kim

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,\frac{1}{\lambda}}\frac{t^n}{n!} = \frac{1}{\lambda}((1+t)^\lambda - 1), \quad (\text{see}[8,15]). \tag{3}$$

From (3), we note that

$$e_\lambda(\log_\lambda(1+t)) = \log_\lambda(e_\lambda(1+t)) = 1+t. \tag{4}$$

In [7], Kim-kim-Lee-Park studied the unsigned degenerate Stirling numbers of the first kind and r-Stirling numbers of the first kind generating function respectively given by

$$\begin{aligned} &\frac{(-\log_\lambda(1-t))^k}{k!} = \sum_{n=k}^{\infty}\begin{bmatrix} n \\ k\end{bmatrix}_\lambda \frac{t^n}{n!}, \\ &\left(\frac{1}{1-t}\right)^r\frac{(-\log_\lambda(1-t))^k}{k!} = \sum_{n=k}^{\infty}\begin{bmatrix} n+r \\ k+r\end{bmatrix}_{r,\lambda}\frac{t^n}{n!}. \quad (\text{see}[7,8,9,10,16]). \end{aligned} \tag{5}$$

The degenerate Stirling numbers of the second kind and r-Stirling numbers of the second kind which are also given by Kim-Kim-Lee-Park

$$\begin{aligned} &\frac{(e_\lambda(t)-1)^k}{k!} = \sum_{n=k}^{\infty}\begin{Bmatrix} n \\ k\end{Bmatrix}_\lambda \frac{t^n}{n!}, \\ &\frac{(e_\lambda(t)-1)^k}{k!}e^r_\lambda(t) = \sum_{n=k}^{\infty}\begin{Bmatrix} n+r \\ k+r\end{Bmatrix}_{r,\lambda}\frac{t^n}{n!}, \quad (\text{see}[7,8,9,10,16]). \end{aligned} \tag{6}$$

The degenerate Genocchi polynomials are defined by

$$\frac{2t}{e_\lambda(t)+1}e^x_\lambda(t) = \sum_{n=0}^{\infty}G_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see}[11,13,14]). \tag{7}$$

when $x=0$, $G_{n,\lambda}(0) = G_{n,\lambda}$ are called degenerate Genocchi numbers.

The degenerate Genocchi polynomials of higher order given by

$$\left(\frac{2t}{e_\lambda(t)+1}\right)^\alpha e^x_\lambda(t) = \sum_{n=0}^{\infty}G^{(\alpha)}_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see}[11,13,14]). \tag{8}$$

when $x=0$, $G^{(\alpha)}_{n,\lambda}(0) = G^{(\alpha)}_{n,\lambda}$ are called degenerate Genocchi numbers of higher-order α.

The degenerate Bernoulli polynomials of higher-order α are defined by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^\alpha e_\lambda^x(t)=\sum_{n=0}^\infty B_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!},\quad(\text{see}[11,12,15,17]).\tag{9}$$

when $x=0$, $B_{n,\lambda}^{(\alpha)}(0)=B_{n,\lambda}^{(\alpha)}$ are called Bernoulli numbers of higher-order α.

In particular, when $\alpha=1$, $B_{n,\lambda}(x)=B_{n,\lambda}^{(1)}(x)$ are called Bernoulli polynomials.

The degenerate Euler polynomials are given by

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^\infty E_{n,\lambda}(x)\frac{t^n}{n!},\quad(\text{see}[11,12,15,17]).\tag{10}$$

when $x=0$, $E_{n,\lambda}(0)=E_{n,\lambda}$ are called Euler number.

The degenerate Euler polynomials of higher order are defined by

$$\left(\frac{2}{e_\lambda(t)+1}\right)^\alpha e_\lambda^x(t)=\sum_{n=0}^\infty E_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!},\quad(\text{see}[11,12,15]).\tag{11}$$

when $x=0$, $E_{n,\lambda}^{(\alpha)}(0)=E_{n,\lambda}^{(\alpha)}$ are called Euler number of higher order.

The r-Stirling Genocchi polynomials given by Roberto-Vernard

$$\begin{aligned}\left(\frac{1}{1+t}\right)^r\frac{2te^{xt}\log(1+t))^k}{k!(e^t+1)}&=\sum_{n=0}^\infty SG_n^1(x;k;r)\frac{t^n}{n!},\\ \frac{2te^{xt}e^{rt}(e^t-1)^k}{k!(e^t+1)}&=\sum_{n=0}^\infty SG_n^2(x;k;r)\frac{t^n}{n!}.\quad(\text{see}[5,6]).\end{aligned}\tag{12}$$

2. Degenerate r-Stirling Genocchi polynomials of the first kind

In this section, we consider the first kind degenerate r-Stirling Genocchi polynomials which are given by

$$\sum_{n=0}^\infty SG_{n,\lambda}^1(x,k,r)\frac{t^n}{n!}=\left(\frac{1}{1-t}\right)^r\frac{2t(-\log_\lambda(1-t))^k}{k!(e_\lambda(t)+1)}e_\lambda^x(t).\tag{13}$$

When $x=0$, $SG_{n,\lambda}^1(0,k,r)=SG_{n,\lambda}^1(k,r)$ are called r-Stirling Genocchi numbers of the first kind.

From (13), we get

$$\begin{aligned}\sum_{n=0}^\infty SG_{n,\lambda}^1(x,k,r)\frac{t^n}{n!}&=\sum_{m=0}^\infty(x)_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^\infty SG_{l,\lambda}^1(k,r)\frac{t^l}{l!}\\&=\sum_{n=0}^\infty\sum_{m=0}^n\binom{n}{m}(x)_{m,\lambda}SG_{n-m,\lambda}^1(k,r)\frac{t^n}{n!}.\end{aligned}\tag{14}$$

Thus, we have the following theorem.

Theorem 1. *For $n,k,r\geq 0$, we have*

$$SG_{n,\lambda}^1(x,k,r)=\sum_{m=0}^n\binom{n}{m}(x)_{m,\lambda}SG_{n-m,\lambda}^1(k,r).$$

From (3) and (13), we get

$$
\begin{aligned}
(15)\qquad \sum_{n=0}^{\infty} SG^{1}_{n,\lambda}(x,k,r)\frac{t^n}{n!} &= \frac{2t(-\log(1-t))^k}{k!(e_\lambda(t)-1)} e^{x}_{\lambda}(t)e^{-r}_{\lambda}(\log_\lambda(1-t))\\
&= \sum_{k=0}^{\infty}(-r)_{k,\lambda}\frac{(\log_\lambda(1-t))^k}{k!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{i=0}^{\infty}SG^{1}_{i,\lambda}(k,r)\frac{t^i}{i!}\\
&= \sum_{i=0}^{\infty}SG^{1}_{i,\lambda}(k,r)\frac{t^i}{i!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{k=0}^{\infty} < r >_{k,\lambda} \sum_{j=k}^{\infty}\begin{bmatrix} j\\ k\end{bmatrix}_{\lambda}\frac{t^j}{j!}\\
&= \sum_{i=0}^{\infty}SG_{i,\lambda}(k,r)\frac{t^i}{i!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{j=0}^{\infty}\sum_{k=0}^{j} < r >_{k,\lambda} \begin{bmatrix} j\\ k\end{bmatrix}_{\lambda}\frac{t^j}{j!}\\
&= \sum_{i=0}^{\infty}SG^{1}_{i,\lambda}(k,r)\frac{t^i}{i!}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}(x)_{m-j,\lambda} < r >_{k,\lambda} \begin{bmatrix} m-l\\ k\end{bmatrix}_{\lambda}\frac{t^m}{m!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{n}{m}\binom{m}{j}(x)_{m-j,\lambda} < r >_{k,\lambda} \begin{bmatrix} m-l\\ k\end{bmatrix}_{\lambda} SG^{1}_{n-m,\lambda}(k,r)\frac{t^n}{n!}.
\end{aligned}
$$

where $< r >_{k,\lambda}$ are degenerate rising factorial sequence.

Thus, by comparing the coefficients on both sides of (15), we have the following theorem.

Theorem 2. *For $n,m,k \geq 0$, we have*

$$
SG^{1}_{n,\lambda}(x,k,r) = \sum_{m=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{n}{m}\binom{m}{j}(x)_{m-j,\lambda} < r >_{k,\lambda} \begin{bmatrix} m-l\\ k\end{bmatrix}_{\lambda} SG^{1}_{n-m,\lambda}(k,r).
$$

From (13), we have

(16)

$$
\begin{aligned}
\sum_{n=0}^{\infty} SG^{1}_{n,\lambda}(x+1,k,r)\frac{t^n}{n!} &= \left(\frac{1}{1-t}\right)^{r}\frac{2t(-\log_\lambda(1-t))^k}{k!(e_\lambda(t)+1)}e^{x+1}_{\lambda}(t)\\
&= \left(\frac{1}{1-t}\right)^{r}\frac{2t(-\log_\lambda(1-t))^k}{k!}e^{x}_{\lambda}(t)\left(1-\frac{1}{e_\lambda(t)+1}\right)\\
&= 2t\sum_{m=k}^{\infty}\begin{bmatrix} m+r\\ k+r\end{bmatrix}_{r,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} - t\sum_{m=k}^{\infty}\begin{bmatrix} m+r\\ k+r\end{bmatrix}_{r,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}E_{l,\lambda}\frac{t^l}{l!}\\
&= 2t\sum_{n=k}^{\infty}\sum_{l=0}^{n}\binom{n}{m}\begin{bmatrix} n-l+r\\ k+r\end{bmatrix}_{r,\lambda}(x)_{l.\lambda}\frac{t^n}{n!} - t\sum_{n=k}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\begin{bmatrix} n-l+r\\ k+r\end{bmatrix}_{r,\lambda}E_{l,\lambda}\frac{t^n}{n!}\\
&= \sum_{n=k+1}^{\infty}\sum_{l=0}^{n-1}2n\binom{n-1}{l}\begin{bmatrix} n-1-l+r\\ k+r\end{bmatrix}_{r,\lambda}(x)_{l.\lambda}\frac{t^n}{n!} - \sum_{n=k+1}^{\infty}\sum_{l=0}^{n-1}n\binom{n-1}{l}\begin{bmatrix} n-1-l+r\\ k+r\end{bmatrix}_{r,\lambda}E_{l,\lambda}\frac{t^n}{n!}.
\end{aligned}
$$

.

By comparing the coefficients of both sides of (16), we have the following theorem.

Theorem 3. *For $n \geq k+1$, we have*

$$
SG^{1}_{n,\lambda}(x+1,k,r) = \sum_{l=0}^{n-1}\left(n\binom{n-1}{l}\begin{bmatrix} n-1-l+r\\ k+r\end{bmatrix}_{r,\lambda}(2(x)_{l,\lambda}-E_{l,\lambda})\right)\frac{t^n}{n!}.
$$

Now, we consider degenerate r-Stirling Genocchi polynomials of the first kind of higher-order α which are given by

$$\sum_{n=0}^{\infty} SG_{n,\lambda}^{(1,\alpha)}(x,k,r)\frac{t^n}{n!} = \left(\frac{1}{1-t}\right)^r \left(\frac{2t}{e_\lambda(t)+1}\right)^\alpha \frac{(-\log_\lambda(1-t))^k e_\lambda^x(t)}{k!}. \tag{17}$$

From (17), we have

$$\begin{aligned}\sum_{n=0}^{\infty} SG_{n,\lambda}^{(1,\alpha)}(x,k,r)\frac{t^n}{n!} &= \sum_{m=0}^{\infty} G_{m,\lambda}^{(\alpha)}(x)\frac{t^m}{m!}\sum_{l=0}^{\infty}\begin{bmatrix} l \\ k \end{bmatrix}_\lambda \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} G_{m,\lambda}^{(\alpha)}(x)\begin{bmatrix} n-m \\ k \end{bmatrix}_\lambda \frac{t^n}{n!}.\end{aligned} \tag{18}$$

Thus, we have the following theorem.

Theorem 4. *For $n,k \geq 0$, we have*

$$SG_{n,\lambda}^{(1,\alpha)}(x,k,r) = \sum_{m=0}^{n}\binom{n}{m} G_{m,\lambda}^{(\alpha)}(x)\begin{bmatrix} n-m \\ k \end{bmatrix}_\lambda.$$

From (17), when $r = \alpha$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} SG_{n,\lambda}^{(1,r)}(x,k,r)\frac{t^n}{n!} &= \left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t)\sum_{m=0}^{\infty}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda \frac{t^m}{m!}\left(\frac{t}{1-t}\right)^r \\ &= \sum_{i=0}^{\infty} E_{i,\lambda}^{(r)}(x)\frac{t^i}{i!}\sum_{m=0}^{\infty}\begin{bmatrix} m \\ k \end{bmatrix}_\lambda \frac{t^m}{m!}\sum_{l=r}^{\infty}\binom{l-1}{r-1}t^l \\ &= \sum_{i=0}^{\infty} E_{i,\lambda}^{(r)}(x)\frac{t^i}{i!}\sum_{j=r}^{\infty}\sum_{m=0}^{j}(j)_m\begin{bmatrix} m \\ k \end{bmatrix}_\lambda\binom{j-m-1}{r-1}\frac{t^j}{j!} \\ &= \sum_{n=r}^{\infty}\sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{j}\binom{j-m-1}{r-1}(j)_m\begin{bmatrix} m \\ k \end{bmatrix}_\lambda E_{n-j,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{19}$$

By comparing the coefficinets on both side of (19), we have the following theorem.

Theorem 5. *For $r = \alpha$, $n, \geq r$, we have*

$$SG_{n,\lambda}^{(1,r)}(x,k,r) = \sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{j}\binom{j-m-1}{r-1}(j)_m\begin{bmatrix} m \\ k \end{bmatrix}_\lambda E_{n-j,\lambda}^{(r)}(x).$$

3. Degenerate r-Stirling Genocchi polynomials of the second kind

Now, we consider the second kind degenerate r-Stirling Genocchi polynomials which are given by

$$\sum_{n=0}^{\infty} SG_{n,\lambda}^{2}(x,k,r)\frac{t^n}{n!} = \frac{2t(e_\lambda(t)-1)^k}{k!(e_\lambda(t)+1)}e_\lambda^{x+r}(t). \tag{20}$$

When $x = 0$, $SG_{n,\lambda}^{2}(0,k,r) = SG_{n,\lambda}^{2}(k,r)$ are called r-Stirling Genocchi numbers of the second kind.

From (20), we have

$$\sum_{n=0}^{\infty} SG^2_{n,\lambda}(x,k,r)\frac{t^n}{n!} = \sum_{m=0}^{\infty}(x+r)_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}G_{l,\lambda}\frac{t^l}{l!}\sum_{i=k}^{\infty}\left\{\begin{matrix} i \\ k \end{matrix}\right\}_{\lambda}\frac{t^i}{i!}$$
$$= \sum_{m=0}^{\infty}(x+r)_{n,\lambda}\frac{t^m}{m!}\sum_{j=k}^{\infty}\sum_{l=0}^{j}\binom{j}{l}G_{l,\lambda}\left\{\begin{matrix} j-l \\ k \end{matrix}\right\}_{\lambda}\frac{t^j}{j!}$$
$$= \sum_{n=k}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(x+r)_{n-j,\lambda}G_{l,\lambda}\left\{\begin{matrix} j-l \\ k \end{matrix}\right\}_{\lambda}\frac{t^n}{n!}. \tag{21}$$

Thus, we have the following theorem.

Theorem 6. *For $n \geq k$, we have*

$$SG^2_{n,\lambda}(x,k,r) = \sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(x+r)_{n-j,\lambda}G_{l,\lambda}\left\{\begin{matrix} j-l \\ k \end{matrix}\right\}_{\lambda}.$$

From (21), we have

$$\sum_{n=0}^{\infty} SG^2_{n,\lambda}(x,k,r)\frac{t^n}{n!} = \sum_{m=0}^{\infty}G_{m,\lambda}(x+r)\frac{t^m}{m!}\sum_{l=0}^{\infty}\left\{\begin{matrix} l \\ k \end{matrix}\right\}_{\lambda}\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}G_{m,\lambda}(x+r)\left\{\begin{matrix} n-m \\ k \end{matrix}\right\}_{\lambda}\frac{t^n}{n!}, \tag{22}$$

and

$$\sum_{n=0}^{\infty} SG^2_{n,\lambda}(x,k,r)\frac{t^n}{n!} = \sum_{m=0}^{\infty}G_{m,\lambda}\frac{t^m}{m!}\sum_{l=0}^{\infty}\left\{\begin{matrix} l+r \\ k+r \end{matrix}\right\}_{r,\lambda}\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}G_{m,\lambda}\left\{\begin{matrix} n-m+r \\ k+r \end{matrix}\right\}_{r,\lambda}\frac{t^n}{n!}. \tag{23}$$

Therefore, we obtain the following theorem.

Theorem 7. *For $n,m \geq 0$, we have*

$$G_{m,\lambda}(x+r)\left\{\begin{matrix} n-m \\ k \end{matrix}\right\}_{\lambda} = G_{m,\lambda}\left\{\begin{matrix} n-m+r \\ k+r \end{matrix}\right\}_{r,\lambda}.$$

From (20), we have

(24)
$$\begin{aligned}
&\sum_{n=0}^{\infty}(r)_{n,\lambda}\frac{t^n}{n!}=\frac{k!(e_\lambda(t)+1)}{2t(e_\lambda(t)-1)^k}\sum_{m=0}^{\infty}SG^2_{n,\lambda}(x,k,r)\frac{t^n}{n!}\\
&=\frac{k!e_\lambda(t)}{2t(e_\lambda(t)-1)^k}\sum_{m=0}^{\infty}SG^2_{m,\lambda}(x,k,r)\frac{t^m}{m!}+\frac{k!}{2t(e_\lambda(t)-1)^k}\sum_{m=0}^{\infty}SG^2_{m,\lambda}(x,k,r)\frac{t^m}{m!}\\
&=\frac{k!}{2t(e_\lambda(t)-1)^k}\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}SG^2_{m,\lambda}(k,r)\frac{t^m}{m!}+\frac{k!}{2t(e_\lambda(t)-1)^k}\sum_{m=0}^{\infty}SG^2_{m,\lambda}(k,r)\frac{t^m}{m!}\\
&=\frac{k!}{2t^{k+1}}\sum_{i=0}^{\infty}B^{(k)}_{i,\lambda}\frac{t^i}{i!}\left(\sum_{j=0}^{\infty}\sum_{m=0}^{j}\binom{j}{m}(1)_{n-m,\lambda}SG^2_{m,\lambda}(x,k,r)+\sum_{m=0}^{\infty}SG^2_{m,\lambda}(x,k,r)\frac{t^m}{m!}\right)\\
&=\frac{k!}{2t^{k+1}}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{m}\binom{j}{m}(1)_{n-m,\lambda}SG^2_{m,\lambda}(x,k,r)B^{(k)}_{i,\lambda}+\sum_{i=0}^{n}\binom{n}{i}SG^2_{n-i,\lambda}(x,k,r)B^{(k)}_{i,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}$$

By multiplying both sides of (24) by $\frac{2t^{k+1}}{k!}$, we have the left hand side of (24) as follows.

$$\tag{25}\begin{aligned}
\frac{2t^{k+1}}{k!}\sum_{n=0}^{\infty}(r)_{n,\lambda}\frac{t^{n+k+1}}{n!}&=\frac{2}{k!}\sum_{n=k+1}^{\infty}(r)_{n-(k+1)}\frac{t^n}{(n-(k+1))!}\\
&=\sum_{n=k+1}^{\infty}2(r)_{n-(k+1),\lambda}\frac{n!}{k!(n-(k+1)!)}\frac{t^n}{n!}\\
&=\sum_{n=k+1}^{\infty}2(k+1)\binom{n}{k+1}(r)_{n-(k+1),\lambda}\frac{t^n}{n!}.
\end{aligned}$$

and multiplying the right hand side of (24) by $\frac{2t^{k+1}}{k!}$, we get

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{m}\binom{j}{m}(1)_{n-m,\lambda}SG^2_{m,\lambda}(x,k,r)B^{(k)}_{i,\lambda}+\sum_{i=0}^{n}\binom{n}{i}SG^2_{n-i,\lambda}(x,k,r)B^{(k)}_{i,\lambda}\right)\frac{t^n}{n!}.$$

By comparing the coefficients of both sides of (25) and (26), we have the following theorem.

Theorem 8. *For $n,k\geq 0$, $n\geq k+1$, we have*

$$\begin{aligned}
&2(k+1)\binom{n}{k+1}(r)_{n-(k+1),\lambda}\\
&=\left(\sum_{j=0}^{n}\sum_{m=0}^{j}\binom{n}{m}\binom{j}{m}(1)_{n-m,\lambda}SG^2_{m,\lambda}(x,k,r)B^{(k)}_{i,\lambda}+\sum_{i=0}^{n}\binom{n}{i}SG^2_{n-i,\lambda}(x,k,r)B^{(k)}_{i,\lambda}\right).
\end{aligned}$$

From (20), we have

(26)

$$\sum_{n=0}^{\infty} SG^{2}_{n,\lambda}(x,k,r)\frac{t^n}{n!} = \frac{2t(-\log_{\lambda}(1-t))^k}{k!(e_{\lambda}(t)+1)}\frac{(e_{\lambda}(t)-1)}{(e_{\lambda}(t)-1)}e_{\lambda}^{x+r}(t)$$
$$= 2(e_{\lambda}^{x+r+1}(t)-e_{\lambda}^{x+r}(t))\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}(-1)^m\lambda^m t^m\binom{\frac{2p}{\lambda}+m-1}{m}\sum_{j=k}^{\infty}\begin{bmatrix} j \\ k \end{bmatrix}_{\lambda}\frac{t^j}{j!}$$
$$= 2\left(\sum_{i=0}^{\infty}(x+r+1)_{i,\lambda}-(x+r)_{i,\lambda}\right)\frac{t^i}{i!}\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}(-1)^m\lambda^m\binom{\frac{2p}{\lambda}+m-1}{m}t^m\sum_{l=k}^{\infty}\begin{bmatrix} l \\ k \end{bmatrix}_{\lambda}\frac{t^l}{l!}$$
$$= 2\left(\sum_{i=0}^{\infty}(x+r+1)_{i,\lambda}-(x+r)_{i,\lambda}\right)\frac{t^i}{i!}\sum_{j=k}^{\infty}\sum_{m=0}^{j}\sum_{p=1}^{\infty}\binom{\frac{2p}{\lambda}+m-1}{m}(-1)^m\lambda^m(j)_m\begin{bmatrix} j-m \\ k \end{bmatrix}_{\lambda}t^j$$
$$= 2\sum_{n=k}^{\infty}\sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{p=1}^{\infty}(n)_j\left((x+r+1)_{n-j,\lambda}-(x+r)_{n-j,\lambda}\right)\binom{\frac{2p}{\lambda}+m-1}{m}(-1)^m\lambda^m(j)_m\begin{bmatrix} j-m \\ k \end{bmatrix}_{\lambda}\frac{t^n}{n!}.$$

Thus, we have following theorem.

Theorem 9. *For $n \geq k$, we have*

$$SG^{2}_{n,\lambda}(x,k,r)$$
$$= 2\sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{p=1}^{\infty}(n)_j\left((x+r+1)_{n-j,\lambda}-(x+r)_{n-j,\lambda}\right)\binom{\frac{2p}{\lambda}+m-1}{m}(-1)^m\lambda^m(j)_m\begin{bmatrix} j-m \\ k \end{bmatrix}_{\lambda}.$$

Now, we consider degenerate r-Stirling Genocchi polynomials of the second kind of higher order α which are given by

$$\sum_{n=0}^{\infty} SG^{(2,\alpha)}_{n,\lambda}(x,k,r)\frac{t^n}{n!} = \left(\frac{2t}{e_{\lambda}(t)+1}\right)^{\alpha}\frac{(e_{\lambda}(t)-1)^k e_{\lambda}^{x+r}(t)}{k!}. \tag{27}$$

when $x=0$ $SG^{(2,\alpha)}_{n,\lambda}(0,k,r) = SG^{(2,\alpha)}_{n,\lambda}(k,r)$ are called degenerate r-Stirling Genocchi numbers of the second kind of higher-order.

From (27), we get

$$\sum_{n=0}^{\infty} SG^{(2,\alpha)}_{n,\lambda}(x,k,r)\frac{t^n}{n!} = \sum_{m=0}^{\infty} G^{(\alpha)}_{m,\lambda}(x+r)\sum_{l=0}^{\infty}\begin{Bmatrix} l \\ k \end{Bmatrix}_{\lambda}\frac{t^l}{l!} \tag{28}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}G^{(\alpha)}_{m,\lambda}(x+r)\begin{Bmatrix} n-m \\ k \end{Bmatrix}_{\lambda}\frac{t^n}{n!}.$$

Thus, we get the following theorem.

Theorem 10. *For $n,k \geq 0$, we have*

$$SG^{(2,\alpha)}_{n,\lambda}(x,k,r) = \sum_{m=0}^{n}\binom{n}{m}G^{(\alpha)}_{m,\lambda}(x+r)\begin{Bmatrix} n-m \\ k \end{Bmatrix}_{\lambda}.$$

From (27), we have

(29)

$$\begin{aligned}
\sum_{n=0}^{\infty} SG_{n,\lambda}^{(2,\alpha)}(x,k,r)\frac{t^n}{n!} &= \left(\frac{2t}{e_\lambda(t)+1}\right)^{\alpha} \frac{(e_\lambda(t)-1)^k e_\lambda^{x+r}(t)}{k!} \\
&= \left(\frac{2t}{e_\lambda(t)+1}\right)^{\alpha} \frac{(e_\lambda(t)-1)^{k+\alpha}}{k!(e_\lambda(t)-1)^{\alpha}} e_\lambda^{x+r}(t) \\
&= (k)_\alpha \left(\frac{2}{e_\lambda(t)+1}\right)^{\alpha} \left(\frac{t}{e_\lambda(t)-1}\right)^{\alpha} e_\lambda^{x}(t) \sum_{m=0}^{\infty} \begin{Bmatrix} m+r \\ k+\alpha+r \end{Bmatrix}_{r,\lambda} \frac{t^m}{m!} \\
&= (k)_\alpha \sum_{l=0}^{\infty} E_{l,\lambda}^{(\alpha)}(x)\frac{t^l}{l!} \sum_{i=0}^{\infty} B_{i,\lambda}^{(\alpha)}\frac{t^i}{i!} \sum_{m=0}^{\infty} \begin{Bmatrix} m+r \\ k+\alpha+r \end{Bmatrix}_{r,\lambda} \frac{t^m}{m!} \\
&= (k)_\alpha \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \binom{j}{l} \begin{Bmatrix} n-j+r \\ k+\alpha+r \end{Bmatrix}_{r,\lambda} E_{l,\lambda}^{(\alpha)}(x) B_{i,\lambda}^{(\alpha)} \frac{t^n}{n!}.
\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 11. *For $n \geq \alpha$, we have*

$$SG_{n,\lambda}^{(2,\alpha)}(x,k,r) = (k)_\alpha \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \binom{j}{l} \begin{Bmatrix} n-j+r \\ k+\alpha+r \end{Bmatrix}_{r,\lambda} E_{l,\lambda}^{(\alpha)}(x) B_{i,\lambda}^{(\alpha)}.$$

4. Conclusion

The generating functions of special polynomials play an important role in finding their characteristics. Furthermore, we can easily approach finding the properties of the degenerate version by grafting this generating function onto the degenerate exponential function. In this paper, we studied the degenerate versions of the r-Stirling Genocchi polynomials and derived their properties. By using other polynomials, such as degenerate Bernoulli polynomials, degenerate Bernoulli polynomials of higher-order α, degenerate Euler polynomials and degenerate Euler polynomials of higher-order, we can express r-Stirling Genocchi polynomials. We will continue this research in the future and try to apply it to other studies.

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